Alignment of Sun, Earth, and Satellite

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The necessary conditions on the launch velocity components of a satellite in order that the satellite may pass through a position of alignment with the earth and sun are derived. In terms of these components and the associated orbital elements, a formula is derived for the calculation of the orbiting time T in transit across the solar disk. Computations on the IBM 7090 computer (shown in graphical form) yield 1) three systems of relations between the launch components of velocity $V_R(e,t,D)$, $V_\theta(e,t,D)$; 2) a system of relations between $t(e)$, $D(e)$; and 3) two systems of relations between aphelion and perihelion radii $R_A(t,D)$, $R_P(t,D)$, in which e,t,D denote eccentricity, time from instant of launch to instant of alignment, and distance from the center of the earth to position of alignment, respectively; D and t are found to be determined uniquely by V_θ and V_θ/V_R, respectively. The data that satisfy the necessary launch conditions are bounded by the inequalities $0.022 < -V_\theta/V_R \leq 0.849$, $0.26 \leq e \leq 0.60$. The corresponding ranges of computed values are obtained: $14 \geq t \geq 1$ days, $50(10)^6 \geq D \geq 0.75\ (10)^6$ km, $10.75 \leq T \leq 12.17$ hr. Typical tabular data (with small t) are $V_R = 39{,}270$ fps, $V_\theta = -1319$ fps, $t = 2$ days, $D = 2(10)^6$ km, $T = 12.17$ hr.

Nomenclature

a,e,E	=	semimajor axis, eccentricity, and true anomaly angle of the satellite orbit, respectively ($\omega = 2\pi - E$)
D	=	distance from earth to satellite's position of solar alignment
$\mathcal{E},S$	=	positions of earth and satellite at instant of alignment with sun's center
$\mathcal{E}_0,S_0$	=	initial position of earth and satellite at time ($t = 0$) of satellite launch
$\mathcal{E}_1,S_1$	=	positions of earth and satellite in alignment with sun's rim
P,A,C	=	perihelion, aphelion, and center of satellite orbit
R_s,θ_s	=	satellite's position coordinates: radial distance and true anomaly angle relative to sun's center
T	=	time interval for satellite's transit across solar disk
t,t_s	=	times (in years measured from the instant of launch of the satellite) corresponding to earth's and satellite's positions $(1, \theta)$, (R_s, θ), respectively
V_θ,V_R	=	initial tangential and radial components of satellite's velocity relative to earth's center $V_\theta = (\dot{\theta}_s)_0 - 2\pi$, $V_R = -(\dot{R}_s)_0$
$1,\theta$	=	position coordinates of earth in circular orbit about sun (radial distance is 1 a.u.; θ is measured from the position of perihelion of the satellite orbit)
$\theta_1,(\theta_1)_s$	=	true anomaly angles corresponding to positions $\mathcal{E}_1$ and S_1, respectively
$\mu = GM$	=	product of universal gravitational constant and mass of sun (in terms of time in years and distance in astronomical units $\mu^2 = 2\pi$)

THE answers to the following questions are the objectives of the present investigation: 1) what are the necessary launch velocity requirements of an earth satellite in order that in a specified time t the position of the satellite becomes aligned with the positions of the earth and sun? 2) what is the possible range of the transit time T of the position of a satellite across the solar disk? and 3) what are the magnitudes of the effects upon T of assumed variations of the launch velocity components? The formulas, which have been derived from a mathematical analysis of the equations of motion and Kepler's equation, form the basis for the computations on an IBM 7090. The computed tabular data appear in Ref. 1. In the present paper, these data are represented graphically by the families of curves which appear in Figs. 1–7. The results of the computations may be used to increase geodetic knowledge if combined with the parallax of the satellite from various stations as the satellite appears to approach the solar disk.

The problem of the relative motions of the sun, earth, and satellite is the restricted three-body problem for which no analytical solution is known. For the present purposes, a first approximation to the conditions of the problem is treated in which a satellite is launched with initial velocity exceeding escape velocity relative to the earth, and thereafter the motions of the earth and satellite are determined solely by the solar gravitational field.

Computations of time and position of satellite solar alignment have been made for a great variety of initial conditions. Typical results obtainable from the tabular data may be described briefly with the aid of the following terminology: the satellite's initial radial and tangential velocity components, relative to the earth, are denoted by V_R and V_θ, respectively. Two important numbers, t and D, are associated with the satellite's position of solar alignment; t is the time interval in days from the satellite's launch position to alignment position, and D is the radial distance in 10^6 km from the earth to the satellite's alignment position. The aphelion radius, perihelion radius, and eccentricity of the satellite orbit are denoted by R_A, R_P, and e, respectively. Figs. 1–4 depict representative curves of one-parameter systems, each curve showing the relation between two variables corresponding to a given constant value of the parameter. Table 1 identifies the variables and curve parameter of each system.

Conditions for Satellite Transit across the Solar Disk

Considering the earth's orbit around the sun to be circular and the orbit of the earth-launched satellite to be elliptic, of eccentricity e and semimajor axis a, one determines that the equations of the earth's and satellite's orbits are, respectively,

$$R = 1 \qquad R_s = a(1 - e\cos E) = \frac{a(1 - e^2)}{1 + e\cos\theta_s} \quad (1)$$

in which the unit of distance is the astronomical unit (1 a.u. = 149.4×10^6 km) and E denotes the eccentric anomaly of the satellite orbit (see Fig. 8). Satellite orbit conditions are sought which will result in a position of alignment of the sun O, the earth E, and satellite S, taking place at a relatively

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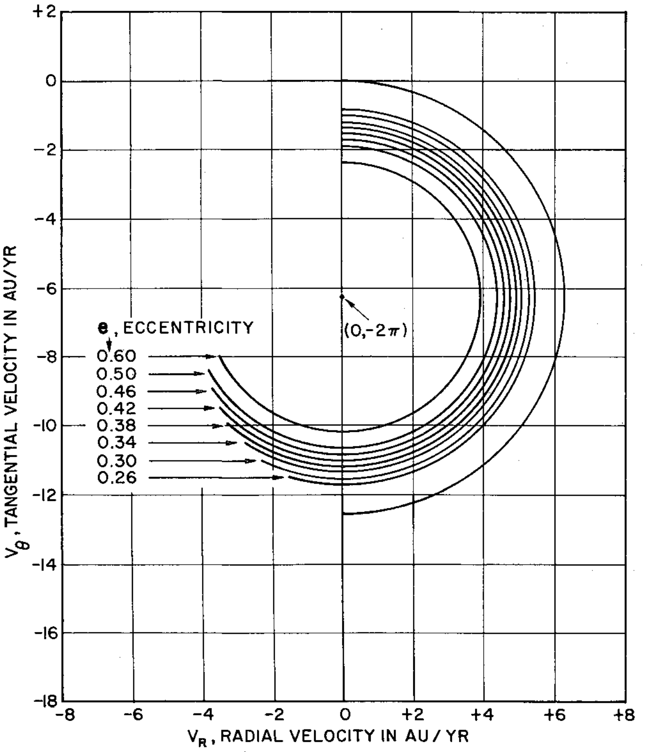


Fig. 1 Possible regions for solar alignment

small distance from the earth compared with the astronomical unit and at a short time after launch compared with the one-year period of the earth's rotation. The analysis of this problem follows.

The relation between eccentric anomaly angle E (see Ref. 2) and true anomaly angle θ_s is given by

$$\theta = 2\tan^{-1}\{[(1+e)/(1-e)]^{1/2}\tan(E/2)\} \quad (2)$$

Regarding the earth and satellite as point masses in the solar system, at launch the coordinates of earth and satellite are given by

$$R = (R_s)_0 = 1 \qquad \theta_0 = \theta_{s_0} \quad (3)$$

Moreover,

$$(R_s)_0 = 1 = a(1 - e\cos E_0) = a(1-e^2)/(1+e\cos\theta_0) \quad (4)$$

The elliptic orbit of the satellite has the center O of the sun as focus, OP as perihelion radius, and OA as aphelion radius (Fig. 8). Corresponding to the satellite position S, the true anomaly angle θ_s is the angle described by a counterclockwise rotation from OP to OS. To define the eccentric anomaly angle E corresponding to the position S, construct the auxiliary circle the diameter of which is the line segment joining aphelion A to perihelion P. The line through S perpendicular to AP intersects the auxiliary circle in the point Q. The eccentric anomaly angle E is the angle described by a counterclockwise rotation from CP to $CQ(0 \le \theta_s \le 2\pi, 0 \le E \le 2\pi)$.

Let time be measured in years and distance in astronomical units, so that in terms of these units

$$\mu^{1/2} = (GM)^{1/2} = 2\pi \quad (5)$$

in which G and M are the universal gravitational constant and mass of the sun, respectively. Moreover, the times (in years, measured from the instant of launch of the satellite) which correspond, respectively, to the earth's and satellite's positions $(1, \theta)$, (R_s, θ) are

$$t = (\theta - \theta_0)/2\pi \quad (6)$$

and

$$t_s = (a^{3/2}/2\pi)[E - E_0 - e(\sin E - \sin E_0)] \quad (7)$$

in which θ is expressible in terms of E by Eq. (2). Alignment occurs if a common value of θ is assumed by the earth's and satellite's positions at the same time, $t = t_s$. Therefore, solutions are sought of the transcendental equation

$$f(E) = a^{3/2}[E - E_0 - e(\sin E - \sin E_0)] - 2\tan^{-1}\left[\left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{E}{2}\right] + 2\tan^{-1}\left[\left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{E_0}{2}\right] = 0 \quad (8)$$

in which a is given by Eq. (4) and θ is defined by Eq. (2).

Solutions of Eq. (8) of the desired type can be found if the following two launch conditions hold: $(\dot\theta_s)_0 < 2\pi$ and $(\dot R_s)_0 < 0$. These inequalities state that the initial angular rate of the satellite is less than that of the earth, and the radial distance R_s is initially decreasing. If the "energy" integral of the equations of motion of the satellite together with Eqs. (1) are differentiated and the results used with the two launch conditions and the known inequalities $E(t) > 0$, $\dot\theta_s(t) > 0$, the following inequalities are obtained:

$$(\dot V_s)_0 > 0 \qquad -1 < \cos E_0 < e \qquad -1 < \cos(\theta_s)_0 < 0 \quad (9)$$

Let ω and v denote the angles defined by the relations

$$\omega = 2\pi - E \qquad v = 2\pi - \theta \quad (10)$$

The inequalities (9) expressed in terms of ω_0, v_0 become

$$-1 < \cos\omega_0 < e \qquad -1 < \cos v_0 < 0 \quad (11)$$

in which $\cos^{-1} e < \omega_0 < \pi$ and $\pi/2 < v_0 < \pi$.

Finally, the solution ω, which corresponds to the solution E of Eq. (8), lies on the interval (ω_0, π).

Equations Connecting Initial Velocity Components with the Satellite's Orbital Elements and Time and Position of Solar Alignment

Let V_θ, V_R denote the initial tangential and radial components of the satellite's velocity relative to the earth's center, defined by

$$\begin{aligned} V_\theta &= (R_s)_0(\dot\theta_s)_0 - \dot\theta = (\dot\theta_s)_0 - 2\pi \\ V_R &= -(\dot R_s)_0 \end{aligned} \quad (12)$$

In view of Eqs. (4, 7, and 10), one may write

$$\frac{V_\theta}{\mu^{1/2}} = [a(1-e^2)]^{1/2} - 1 = \left(\frac{1-e^2}{1-e\cos\omega_0}\right)^{1/2} - 1 \quad (13)$$

$$\frac{V_R}{\mu^{1/2}} = \frac{e\sin\omega_0}{(1-e\cos\omega_0)^{1/2}} \quad (14)$$

To determine the times and positions when the satellite can be in alignment, one first computes the real root $\omega(\omega < \omega_0)$ nearest to ω_0 of the equation

$$\omega_0 - \omega + e(\sin\omega - \sin\omega_0) + 2[1 - e_0\cos\omega_0]^{3/2} \times \left\{\tan^{-1}\left[\left(\frac{1+e}{1-e}\right)^{1/2}\tan(\omega/2)\right] - \tan^{-1}\left[\left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{\omega_0}{2}\right]\right\} = 0 \quad (15)$$

in which ω_0, e are selected to satisfy inequalities (11), Eq. (15) being the result of transforming Eq. (8) by means of the substitutions (10). The time and position coordinates of the

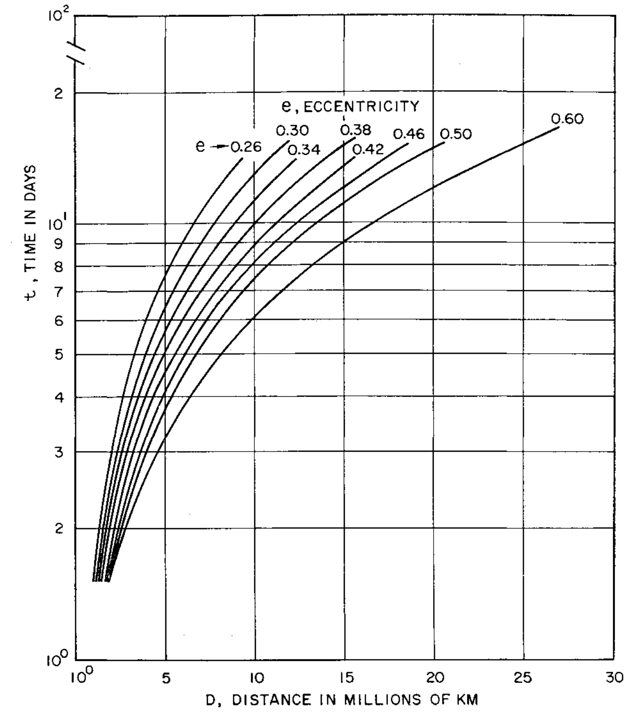


Fig. 2 Time vs satellite radial distance for solar alignment

satellite corresponding to the roots of Eq. (15) [in view of Eqs. (1, 2, 4, 7, and 8)] are given by

$$\mu^{1/2}t = \frac{\omega_0 - \omega - e(\sin\omega_0 - \sin\omega)}{(1 - e\cos\omega_0)^{3/2}} \qquad (16)$$

$$R_s = \frac{1 - e\cos\omega}{1 - e\cos\omega_0} \qquad \theta_s = 2\pi - v \qquad (17)$$

where

$$v = 2\tan^{-1}\{[(1+e)/(1-e)]^{1/2}\tan(\omega/2)\}$$

and

$$0 < \omega < \pi \qquad 0 < v < \pi$$

Also, from Eqs. (1) and (10), the equation for semimajor axis a is obtained:

$$a = 1/(1 - e\cos\omega_0) \qquad (18)$$

The formulas for aphelion radius R_A and periphelion radius R_P consequently are

$$R_A = a(1+e) = (1+e)/(1 - e\cos\omega_0)$$
$$R_P = a(1-e) = (1-e)/(1 - e\cos\omega_0) \qquad (19)$$

The distance D from the earth to the satellite position, in view of Eqs. (17) and (18), is given by

$$D = 1 - R_s = [e(\cos\omega - \cos\omega_0)]/(1 - e\cos\omega_0) \qquad (20)$$

In this section, as in the preceding section, the unit of distance is the astronomical unit (1 a.u. = 149.4 × 10^6 km), and the unit of time is one year. In this system of units $(GM)^{1/2} = 2\pi$, where G is the universal gravitational constant and M is the mass of the sun.

It readily follows from Eqs. (13) and (14) that

$$(V_\theta + 2\pi)^2 + V_R^2 = 4\pi^2(1 + e\cos\omega_0) \qquad (21)$$

Also, from the first inequalities of (17), one has

$$1 - e < 1 + e\cos\omega_0 < 1 + e^2 \qquad (22)$$

A combination of Eq. (21) with the inequalities (22) yields

$$4\pi^2(1-e) < (V_0 + 2\pi)^2 + V_R^2 < 4\pi^2(1+e^2) \qquad (23)$$

Thus, the implications of the two launch conditions are that V_R and V_θ satisfy the inequalities (23). Interpreted geometrically, the inequalities (23) state that permissible values of V_R, V_θ are coordinates of points lying within an annulus whose center is the point $(-2\pi, 0)$, and whose bounding circles have radii $2\pi(1-e)^{1/2}$ and $2\pi(1+e^2)^{1/2}$. Moreover, from (11, 13, and 14), it is seen that V_R, V_θ satisfy the inequalities (see Fig. 1)

$$0 < V_R < 2\pi e/(1-e^2)^{1/2} \qquad (24)$$

$$2\pi - [1 + (1-e)^{1/2}] < V_\theta < 0 \qquad (25)$$

Corresponding to a given eccentricity e, the values of V_R, V_θ (Fig. 1) which produce satellite solar alignment in the interval $\pi < \theta_s < 2\pi$ are the coordinates of a point of the quadrant $V_R > 0$, $V_\theta < 0$ which lies within the region bounded by the concentric circles whose radii are $2\pi(1-e)^{1/2}$ and $2\pi(1+e^2)^{1/2}$, and whose common center is the point $(0, -2\pi)$.

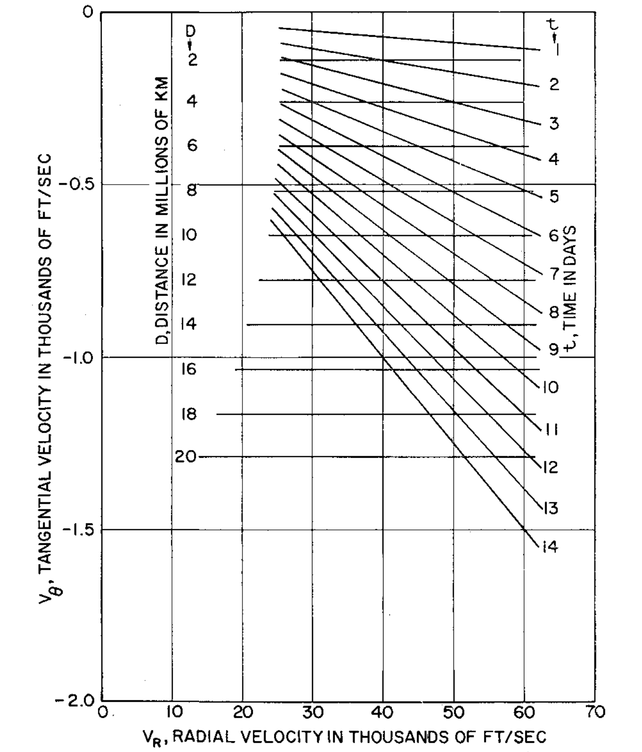


Fig. 3 Satellite tangential velocity vs radial velocity for solar alignment (at given time and distance)

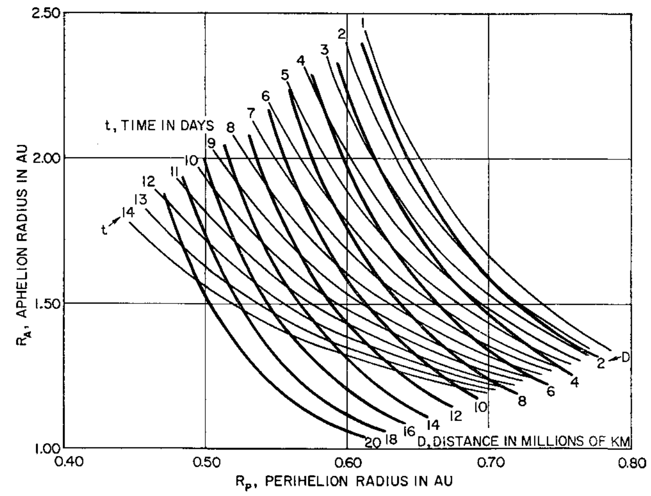


Fig. 4 Satellite aphelion radius vs perihelion radius for solar alignment (at given time and distance)

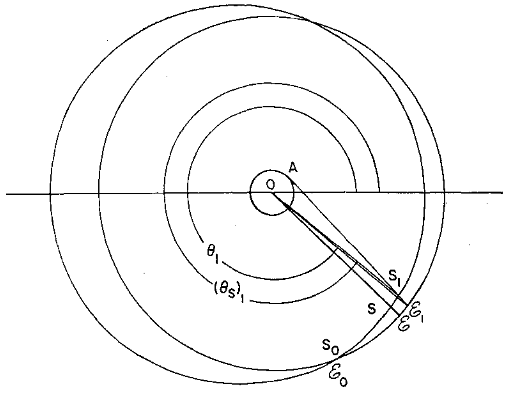


Fig. 5 Geometry for satellite transit across solar disk

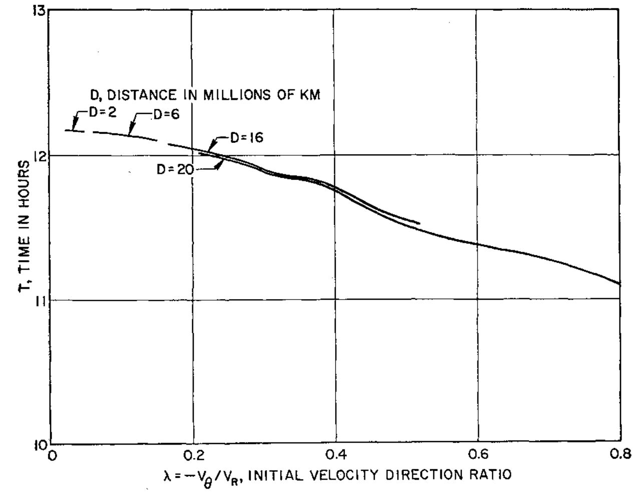


Fig. 6 Satellite solar transit time vs initial velocity direction ratio for solar alignment

Numerical and Graphical Solutions Related to Solar Alignment

A numerical analysis procedure has been developed to calculate the real root $\omega(\omega < \omega_0)$ nearest to ω_0 of the transcendental Eq. (15). The iterative procedure thus developed has been programmed and coded, and each root ω corresponding to a pair of values (e,ω_0) among hundreds of such selected pairs has been computed on the IBM 7090 digital computer. The computations of the associated values of t, V_θ, V_R, R_S, θ_S, D, R_P, R_A were incorporated into the program. Suitable conversion factors were introduced to convert time interval t to days, V_θ, V_R to feet per second, D to kilometers. The systems of curves appearing in Figs. 2–4 are accurate graphical representations of the computed data.

The time t of the satellite's solar alignment position (Fig. 2) is given as a function of the distance D to this position for various eccentricities (t is given in days from instant of launch).

At the point of intersection of the curves of the two systems (Fig. 3), the parameter values t and D are the alignment time and distance, and the coordinates of the point are the corresponding initial velocity components V_R and V_θ (time: 1 to 14 days; distance: 2 to 20 km $\times$ 10^6).

Table 1 Variables and curve parameters for Figs. 1–4

Figure	Variables	Curve parameters
1	V_R, V_θ	e
2	D, t	e
3	V_R, V_θ	t, D
4	R_A, R_P	t, D

At the point of intersection of the curves of the two systems (Fig. 4), the coordinates of the point are the initial velocity components V_R and V_θ, and the alignment time t and distance D are the corresponding parameter values.

Conclusions

The graphs of Fig. 3 are sensibly straight lines; if extended, those related to t would pass through the origin, whereas those related to D are parallel to the axis of V_R. Therefore, the time t of solar alignment is determined uniquely by the value of the ratio V_θ/V_R, whereas the distance D is determined uniquely by the value of V_θ. Moreover, a t = const graph intersects a D = const graph in the point whose coordinates are the initial velocity components V_R, V_θ which give rise to solar alignment in time t at distance D. For example, solar alignment in $t = 2$ days at a distance $D = 2(10^6)$ km would result from initial satellite velocity components $V_R = 39{,}270$ fps, $V_\theta = -1319$ fps. The tabular data, however, extend through the intervals between the following two sets of corresponding data: 1) $t = 7(10)^{-5}$ days, $D = 76$ km, $V_R =$ 36, 134.24 fps, $V_\theta = -1.94151$ fps; and 2) $t = 15.0733$ days, $D = 28.12485$ (10^6) km, $V_R = 66{,}689.514$ fps, $V_\theta = -47{,}365.311$ fps.

Computations of Possible Time Intervals for a Satellite's Transit across the Solar Disk

To derive a general formula for the calculation of the time interval of transit of a satellite across the solar disk, the geometry of the relative positions of the earth, satellite, and sun in the neighborhood of the solar alignment positions is needed. This geometry is illustrated in Fig. 5, in which, for the sake of visual clarity, certain relatively small quantities are magnified.

Let θ_s, $(\theta_s)_1$ and ω, ω_1 denote the true anomaly angles and eccentric anomaly angles, respectively, of the satellite positions S, S_1, and let θ, θ_1 denote the true anomaly angles of the corresponding earth positions $\mathcal{E}$, $\mathcal{E}_1$.

Let quantities $\Delta\omega$, $\Delta\theta_s$, $\Delta\theta$ be defined by the relations

$$\Delta\omega = \omega_1 - \omega \qquad \Delta\theta_s = (\theta_s)_1 - \theta_s \qquad \Delta\theta = \theta_1 - \theta \tag{26}$$

The acute angle $O\mathcal{E}_1A$, which will be denoted by α, is approximately equal to $\frac{1}{4}°$.

Since θ, θ_s correspond to solar alignment positions of earth and satellite, θ is equal to θ_s; therefore

$$<\mathcal{E}_1OS_1 = \Delta\theta_s - \Delta\theta = (\theta_s)_1 - \theta_1 \tag{27}$$

Also, since the earth's orbit is assumed to be circular (radius = 1 a.u.), $|O\mathcal{E}_1| = 1$. An application of the sine law to the triangle $O\mathcal{E}_1S_1$ yields the relation

$$\sin\alpha/OS_1 = \sin(\alpha + \Delta\theta_s - \Delta\theta) \tag{28}$$

The solar alignment positions of earth and satellite are denoted by $\mathcal{E}$ and S, respectively (Fig. 5). After a lapse of time Δt, the positions of earth $\mathcal{E}_1$ and satellite S_1 are aligned with the sun's rim, i.e., the line $\mathcal{E}_1S_1$ is tangent to the surface of the sun at the point A.

In terms of ω, one has

$$OS_1 = R_S(\omega + \Delta\omega) = R_S + R_S'\Delta\omega + \tfrac{1}{2}R_S''(\Delta\omega)^2 + \ldots \tag{29}$$

$$\Delta\theta_S = \theta_S'\Delta\omega + \tfrac{1}{2}\theta_S''(\Delta\omega)^2 + \ldots \tag{30}$$

$$\Delta\theta = 2\pi\Delta t = -a^{3/2}\{\Delta\omega - 3[\sin(\omega + \Delta\omega) - \sin\omega]\} \simeq -a^{1/2}[R_S\Delta\omega + \tfrac{1}{2}R_S'(\Delta\omega)^2] \tag{31}$$

in which the prime denotes differentiation with respect to ω. Substituting from (30–32) into (28), neglecting terms of order

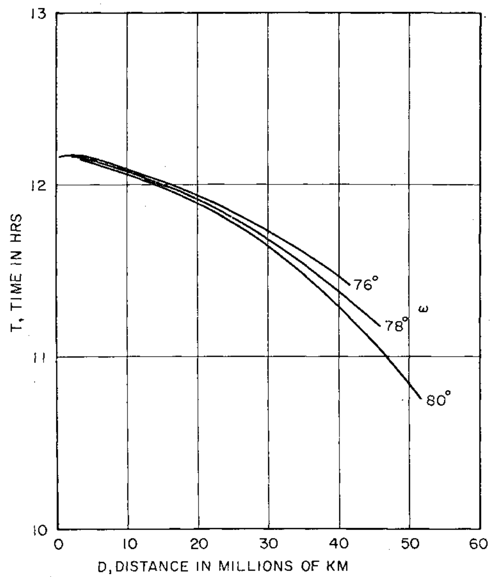


Fig. 7 Satellite solar transit time vs solar alignment distance

$(\Delta\omega)^3$, leads to the following quadratic equation in $\Delta\omega$:

$$\{(\sin\alpha/2)[R_S'' - R_S(\theta_S' + a^{1/2}R_S)^2] + \cos\alpha[R_S'(\theta_S' + a^{1/2}R_S) + (R_S/2)(\theta_S'' + a^{1/2}R_S')]\}(\Delta\omega)^2 + \{R_S' \sin\alpha + R_S \cos\alpha(\theta_S' + a^{1/2}R_S)\}\Delta\omega + (R_S - 1)\sin\alpha = 0 \quad (32)$$

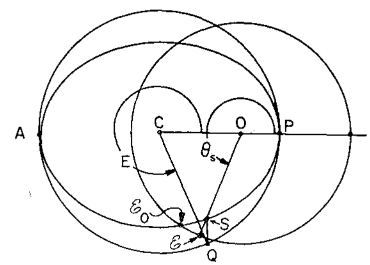
This equation can be expressed in the following form, the coefficients of which are functions of a, e, α, R_S, ω by substituting the known values for R_S', R_S'', θ_S', θ_S'':

$$\left(\frac{\sin\alpha}{2}\left\{ae\cos\omega - R_S\left[a^{1/2}R_S - \frac{a(1-e^2)^{1/2}}{R_S}\right]^2\right\} + \frac{ae\cos\alpha\sin\omega}{2}\left[3a^{1/2}R_S - \frac{a(1-e^2)^{1/2}}{R_S}\right]\right)\Delta\omega^2 + \{ae\sin\alpha\sin\omega + \cos\alpha\,[a^{1/2}R_S^2 - a(1-e^2)^{1/2}]\}(\Delta\omega) + (R_S - 1)\sin\alpha = 0 \quad (33)$$

Finally the time T of transit of the satellite across the solar disk is given by

$$T = 2\Delta t = -(a^{3/2}/\pi)\{\Delta\omega - e[\sin(\omega + \Delta\omega) - \sin\omega]\} \quad (34)$$

Fig. 8 Geometry for solar alignment of earth and satellite



in which $\Delta\omega$ is the negative root of Eq. (33). In this form of the equation, T is expressed in years and the length a in astronomical units.

The roots of Eq. (33) and the associated values of T have been computed for a wide range of parameter values a, e, R_S, ω taken from the tables already computed (Tables I, II, III of Ref. 1). The results enable one to determine the solar transit times T which correspond to wide ranges of initial values of V_R, V_θ and also to examine the variations of T as functions of alignment position (D, ω). Typical results are presented in Figs. 6 and 7.

The satellite's time T in transit across the solar disk is presented as a function of λ for various values of D (Fig. 6).

The satellite's solar transit time T (Fig. 7) is presented as a function of D for various values of ω. (The angle ω is defined in terms of the eccentric anomaly angle E of the satellite's solar alignment position by the relation $\omega = 2\pi - E$.)

Conclusions

Corresponding to the input data selected, possible values for T were found to range from 10.75 to 12.17 hr. It is interesting to observe that the variation of T with V_θ/V_R is nearly independent of D, and its variation with D is nearly independent of ω. Tables VI and VII of Ref. 1 from which Figs. 6 and 7 were drawn include ranges of initial data which gives rise to a variation of alignment distance D from 0.75 (10^6) to $50(10^6)$ km and a variation of alignment angle ω from 50° to 100°.

References

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